$$G_{\mu\nu} = -8\pi T_{\mu\nu}$$

A MOST INCOMPREHENSIBLE THING

Notes towards a very gentle introduction to the mathematics of relativity

- A self-study guide to special and general relativity
- From simple functions to relativistic cosmology
- With numerous fully solved problems

Peter Collier

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A Most Incomprehensible Thing: Notes Towards a Very Gentle Introduction to the Mathematics of Relativity

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Notes Towards a Very Gentle Introduction to the Mathematics of Relativity

Peter Collier

Incomprehensible Books

For Anne Redfarn, xxx

Preface

This book is written for the general reader who wishes to gain a basic but working understanding of the mathematics of Einstein's theory of relativity, one of the cornerstones of modern physics.

I must have been eleven or twelve years old when I treated myself to a paperback layperson's introduction to the theory. I enjoyed maths and science at school, vaguely knew that relativity was a difficult but important theory, and was curious to find out more. Being a popular guide, there was very little mathematics in the text, but there were a sprinkling of simple equations from the special theory of relativity (as we'll see, Einstein proposed two theories: special relativity and general relativity). These formulations described relativistic phenomena such as time dilation and length contraction - nice, straightforward equations that even I could understand. I remember wondering what all the fuss was about if those equations were all there were to relativity. The complexity of Einstein's theory had obviously been exaggerated. I read and enjoyed the book and mentally ticked off the theory as something else I had mastered, like long division or factorisation.

I was hopelessly wrong, of course. I'd made the mistake of confusing popular science with the real thing, and believing I'd grasped the theory of relativity when in reality I'd barely scratched its surface. It took me a long time to realise that those equations, though perfectly valid, were but a tiny part of a much larger, more complex and wider ranging description of the physical world.

Although my formal education in maths and physics ended some time ago, I've continued to enjoy dabbling around in the subjects. During the winter of 2010-11, with time on my hands, I came across the excellent series of YouTube general relativity lectures by Professor Leonard Susskind [30] of Stanford University. Much of what the professor said went over my head, but after watching the series, I found myself intrigued with the language of spacetime, gravitation, metrics, tensors and black holes. For the second time in my life I resolved to try to teach myself relativity. This time around I was more aware of the size and shape of the challenge. Plus, of course, I now had access to the resources of the internet (and more pocket money for textbooks!). My goal was to move beyond the popular expositions of relativity and get to grips with the underlying mathematics, the beating heart of the theory (to paraphrase Euclid, there is no royal road to relativity – you have to do the maths). And so my adventure began. For the next twelve months I was obsessed, with almost every spare moment, at home and at work, spent poring over books and websites. It was hard work but great fun, and fortunately my partner was supremely patient with my new infatuation.

Thus also began my quest for my ‘ideal’ relativity textbook. There are of course various maths-lite popular guides, such as the one I'd read as a boy. Plus there are technically demanding undergraduate and higher-level textbooks. The popularisations may be entertaining, but by excluding the maths they can only give a cursory understanding of the subject. The mathematically rigorous texts, on the other hand, are unreadable for the non-specialist. I was looking for something in-between, a Goldilocks volume pitched just at my level, neither too easy nor too difficult. I didn't want to take a degree in physics, but I did want to get to grips with the essential mathematics of relativity.

I never found my ideal volume but instead had to make use of many different sources, winkling out bits of useful information here and there, struggling to fit the different pieces of the jigsaw into a coherent whole. Along the way, it dawned on me that if I were looking for a self-contained, introductory, mathematical text on relativity, others might be as well; after all, it is one of the most important theories in physics. The book I had in mind would assume little prior mathematical knowledge (even less than my own patchy sixth-form/high school maths if it were to be suitable for the general reader). It would therefore need to begin with a crash course in foundation maths. To give the kind of meaningful mathematical understanding I sought, it would try wherever possible to give the relevant derivations in full, even at the risk of stating what appears to be the blindingly obvious to the more mathematically savvy reader. And it would contain numerous fully worked problems, because in my experience, seeing how the mathematics is used in practice is the best way to understand it. Oh, and it would be written in a user-friendly style with lots of helpful diagrams and pictures. And - given the high cost of many textbooks - it would be inexpensive verging on the downright cheap.

What you see in front of you is my attempt to write such a book, the accessible teach-yourself study aid I would have liked to have got my hands on when I first seriously started to learn Einstein's theory - an introduction to the mathematics of relativity for the enthusiastic layperson. By ‘layperson’ I mean someone with a minimal mathematical background, though obviously there are no penalties if yours exceeds that - just skip what you know. ‘Enthusiastic’ suggests this may not be an easy journey, but one that demands some degree of commitment and effort from the reader. Physicists may believe that the language of Nature is mathematics, but in the case of general relativity she might have made that language a touch easier to learn for the average *Homo sapiens*. Even at our basic level, if we really want to understand what's going on in spacetime, we need to tackle the delights of things such as tensors, geodesics and, of course, the Einstein field equations.

There's no escaping the fact that it's not easy learning a technically demanding subject such as relativity on your own. Away from college or university, the self-studier ploughs a lonely furrow, with no structured coursework, lectures or interaction with tutors and fellow students. However, the pursuit of education and understanding for its own sake is an admirable goal and deserves every encouragement. We should all have the chance to appreciate - in the words of Matthew Arnold - ‘the best that has been said and thought in the world.’ This book tries to offer a helping hand to such intrepid seekers of truth. In a world overflowing with trivia, irrationality and nonsense, relativity is the genuine article, a challenging but fundamentally important scientific theory.

I would suggest that for most readers the best way to approach this book is as an ultra-relaxed marathon rather than a sprint. We cover a lot of material, from elementary mathematics to tensor calculus, so give yourself plenty of time to thoroughly understand what's being presented. You're bound to get stuck sooner or later. Indeed, you will probably grind to a halt more than once - I know I did. Try not to be discouraged. Remember, Einstein was a genius, yet it took even him ten years to formulate general relativity. Personally, I found that if I hit a brick wall, instead of struggling on, it helped to put the book I was studying to one side and take a break. Time for reflection, working on something else, maybe a good night's sleep were often all that was necessary for comprehension to dawn. If you are still baffled, you could seek enlightenment via one of the excellent online physics/maths forums (three are listed in the bibliography - [20], [25] and [26]). Of course, the ideal is to finish the book, but if you don't, at least try to have fun getting through as much as you can. Thankfully, you don't need to be a genius to appreciate the wonders of relativity or to ponder the strange, mysterious world that the theory so accurately describes.

- I've made every effort to ensure that there are no errors in this book. However, mistakes can happen. A current list of errata can be found at http://amostincomprehensiblething.wordpress.com. If you find an error that does not appear on that list, please e-mail it to incomprehensiblething@gmail.com.
- Comments, feedback? Email the author at the same address.
- Apologies that much of the mathematics in the text is in the form of non-scalable images. I originally wrote the book using LyX, a maths-friendly (and much more) document processor based on LaTeX. LyX produces beautiful pdf documents with great looking mathematics (see the paperback print edition), but trying to convert it to a Kindle/EPUB file was fiendishly difficult. Given my non-existent coding skills, the best solution I came up with was to export LyX to XHTML, with the equations converted to images. Where possible, I then converted the simpler equations to html by hand, but the majority remain as images. Maybe one day, if LyX improves its exporting capabilities, I'll be able to bring out another edition with all the maths as beautiful flowing text.
- Following on from the previous point, I would advise readers who intend reading this ebook primarily on a Kindle e-reader to check they are comfortable with the legibility of the smaller, more intricate equation images. Unfortunately, at the time of writing, smaller images are not zoomable on certain Kindles. To avoid disappointment, it would therefore be a good idea to skim through the ebook soon after purchase. If you aren't happy with the size of the images, you can either (a) read the book on a larger screened device (tablet, PC, laptop etc) using one of Amazon's excellent free Kindle apps, or (b) return ebooks you bought from the Kindle Store for a refund (Amazon must receive your request within seven days of the date of purchase).

Note on the revised edition

I have been most pleasantly surprised (thrilled, to be honest) by the interest and enthusiasm shown towards my book since it was first published two years ago. Readers of all ages and from a wide variety of backgrounds have clearly relished the opportunity of trying to understand Einstein's wonderful theories. As a general point, I believe there is no little demand out there for books that, without dumbing down, attempt to make complex subjects intelligible to the general reader. (If anyone is willing to pick up the baton, I know from readers' feedback that many would appreciate a ground up guide to quantum mechanics.)

However, despite the most careful preparation, it was inevitable that a work of this scope and complexity would contain errors, many of which have been spotted and emailed to me by sharp-eyed readers. I must of course apologise for these mistakes and also thank all those who took the trouble to contact me. Without their input this book would be a very much inferior shadow of its present self. Over time, that trickle of emails slowly reduced to an intermittent drip, and - dare I say it? - for a while now my inbox has actually been free of such tidings. Hence my decision to publish a revised edition, which contains no new material, I'm afraid, but does include all those corrections and clarifications to the first edition listed in the online errata sheet. (This revised ebook edition contains the same text as the second paperback edition.)

Introduction

The most incomprehensible thing about the world is that it is at all comprehensible.
ALBERT EINSTEIN

During the first few years of the twentieth century, Albert Einstein (Figure 0.1) revolutionised our understanding of the physical world. In 1905 he proposed his special theory of relativity, which fatally undermined long-standing scientific and common sense assumptions about the nature of space and time. Simultaneity, for example: the new theory meant that two events happening at the same time for one observer might well occur at *different* times for another. Henceforth, space and time could no longer be regarded as separate and absolute quantities. Instead they merged - the theory and all available evidence demanded they merge - into a new single entity called spacetime. Furthermore, matter and energy were also joined, in the shape of one of the most famous equations in physics: $E = mc^2$.

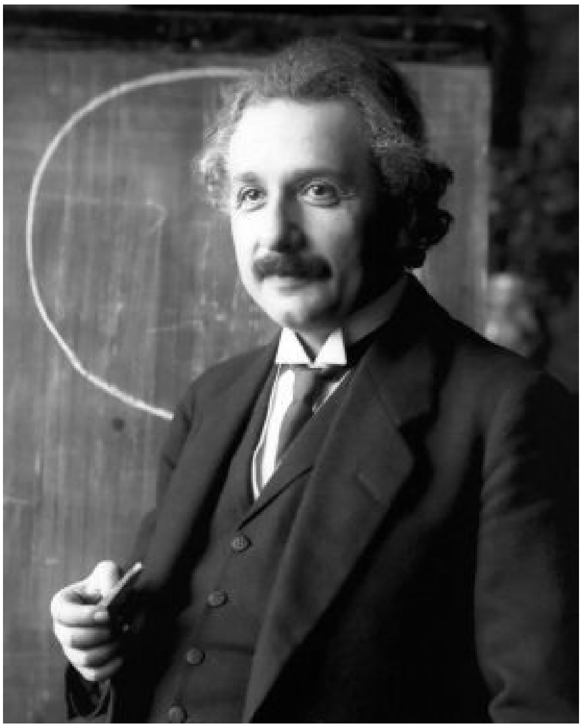

Figure 0.1:
Albert Einstein (1879–1955) - photographed in 1921.

This was radical stuff, but much more was to come. Special relativity deals with the motion of objects and of light in the absence of gravity. For over two hundred years Newtonian gravitation had proved itself a theory of astonishing accuracy. And it still is, allowing the precise calculation of the motion of a falling cup or an orbiting planet. But though of immense practical use, Newton's theory isn't compatible with special relativity (Newtonian gravity is instantaneous; special relativity imposes a natural speed limit - the speed of light). It took Einstein a further ten years to reconcile gravity and special relativity. That synthesis, his general theory of relativity, was published in 1916. General relativity explains gravity as an effect of the bending of spacetime in the vicinity of a massive object. The general theory describes phenomena as diverse as non-Newtonian deviations in planetary orbits, gravitational time dilation, and gravitational bending and redshift of light. It has been used to predict the existence and properties of black holes and is at the heart of modern cosmology -- the study of the history and structure of the universe. Necessarily, given the appropriate conditions, general relativity is smart enough to reduce to both special relativity and Newtonian theory.

Special and general relativity can be summarised as follows:

- The speed of light in a vacuum has the same value for all uniformly moving observers.
- Mass and energy curve spacetime.
- Mathematically, spacetime can be represented by a curved space that is locally flat.
- An equation called a metric describes the curvature of such a space. The metric will vary from region to region depending how the space curves.
- Light and free particles follow paths, called geodesics, through spacetime determined by how the spacetime is curved.
- In the appropriate circumstances general relativity should approximate both to special relativity and to Newtonian gravitation.
- The laws of physics must take the same form in all coordinate systems.

Or, more succinctly, the famous quotation from physicist John Archibald Wheeler (1911–2008) states that:

> *‘Matter tells space how to curve.*
> *Space tells matter how to move.’*

The astrophysicist Kim Griest [12] asserted the fundamental importance of general relativity when he wrote:

> *‘Most physicists don't study general relativity [GR] because it only differs from Newton's gravity theory and from special relativity in a few cases. But GR is Nature's choice - whenever GR differs from Newton, GR has been shown to be right. It is how Nature actually works, and requires a radical rethinking of physical reality.’*

In short, relativity is a triumph of human reason, and as such deserves the widest possible audience. However, to really understand even the basics of the theory we have to tackle some quite challenging mathematics, and that is what we are going to attempt to do.

Chapter 1 sets the ball rolling by introducing the necessary foundation mathematics needed to progress through the rest of the book, from the basic definition of a function, through calculus and simple vectors, to our first metric tensor. For those with little mathematical background this chapter will be a baptism of fire. Emerging from the flames, we advance invigorated into Chapter 2. I don't see how it's possible to get to grips with relativity without understanding at least some of the physics it supplanted. To that end, this chapter comprises a brisk discussion of Newtonian mechanics, with more time spent, deservedly I trust, on Newton's wonderful theory of gravitation, including a neat little detour on how to plot the orbit of a hypothetical planet around the Sun. In Chapter 3 we move on to special relativity and the strange world of Minkowski spacetime, including the counter-intuitive phenomena of time-dilation and length contraction. After developing our spacetime intuition with the geometrical assistance of spacetime diagrams we progress to a more algebraic approach using the Lorentz transformations. We end this chapter by looking at how special relativity reformulates the laws of mechanics.

The next three chapters develop the mathematics that underpins general relativity. First, a brief introduction to the concept of the manifold and the all important metric tensor $g_{\mu\nu}$. Next we look at vectors, one-forms and tensors in order to ease us into the mathematics of curvature, including connection coefficients, parallel transport of vectors, geodesics and the Riemann curvature tensor. Chapter 7 pulls these various strands together to take us to the star of the show: the Einstein field equations. On the way we meet the equivalence principle (Einstein's ‘happiest thought’), geodesics in spacetime and the energy-momentum tensor. In Chapter 8 we see how, given appropriate non-relativistic conditions, the equations of general relativity approximate to the ultra-successful formulations of Newtonian mechanics. The next chapter introduces the Schwarzschild solution, the first and most important exact solution to the Einstein field equations. This solution provides a good approximation to the gravitational field of slowly rotating bodies such as the Sun and Earth. We derive the Schwarzschild solution and use it to discuss the four classical tests of general relativity. The Schwarzschild solution can be used to predict and describe the simplest type of black hole, which we do in Chapter 10. Here we investigate the weird nature of time and distance as experienced by an observer unfortunate enough to fall into a black hole, compared to how that same journey appears to a distant observer.

The final chapter is a brief introduction to relativistic cosmology. We start with four key observed properties of the universe, including the cosmological principle - the assumption that on a very large scale the universe looks the same to all observers, wherever they are. The Robertson-Walker metric and the Friedmann equations together establish the theoretical framework that then allows us to discuss several simple cosmological models and gain insight into the history and evolution of our own universe.

Note, in this book, you are going to (pun alert) see a lot of *c*, the symbol that denotes the speed of light in a vacuum and which (approximately) equals 300,000,000 metres per second. Be aware that in relativity life is often made simpler by defining *c* as being equal to 1. We sort of do this ourselves when we start using spacetime diagrams in special relativity. This is perfectly legitimate, all we are doing is playing around with units of measurement. Some authors then take this practice a step further and go on to omit c from their equations. For example, the Lorentz factor (3.4.1), which in this book is given by

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}},$$

would become (assuming $c = 1$)

$$\gamma = \frac{1}{\sqrt{1-v^2}}.$$

To the learner, it's not obvious that these two equations mean the same thing (which they do). There's scope for confusion here, which is the reason I include c in all the relevant equations in this book.

Also note that whenever we refer to ‘light’ we aren't of course only referring to the narrow range of the electromagnetic spectrum visible to the human eye, but to all electromagnetic radiation, such as gamma rays, X-rays, visible light, radio waves, etc.

1 Foundation mathematics

Do not worry about your difficulties in mathematics. I can assure you mine are still greater.
ALBERT EINSTEIN

1.1 Introduction

In order to make sense of what follows, we need to introduce some essential maths. There's quite a lot of it, and those with a limited mathematical background may find this chapter somewhat of a challenge. Furthermore, lack of space permits only a brief discussion of a wide range of topics. So although I've tried to include as much information as seems necessary, you may need to forage elsewhere for additional insights and support (see, for example, [21] and [24]). But take heart. With enthusiasm and perseverance we can now begin to familiarise ourselves with a broad sweep of fundamental mathematical ideas. By the end of this chapter we'll have met functions, exponents, coordinate systems, calculus, vectors, matrices and more, including our very first metric tensor, the central object of study in relativity.